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1971 J. Phys. A: Gen. Phys. 4 597

(http://iopscience.iop.org/0022-3689/4/5/001)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Generalized field equations in general relativity

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MS. received 30*th March* 1971

Abstract. Generalizations of Einstein's field equations in general relativity are derived by considering variational principles in which the components of an ennuple, or tetrad, are the quantities which undergo variation. The conservation law for the field equations is established and the weak field approximations discussed. The equations are solved by the method developed in a previous article. Many solutions are obtained, all of which are inferior to the Schwarzschild solution.

1. Introduction

In a previous article (Tupper and Phillips 1970, to be referred to as I), we have suggested a method of solving alternative field equations in general relativity based on the use of ennuple or tetrad vectors. The particular field equations which were the subject of that investigation were those proposed by Kilmister (1967) in an attempt to overcome some of the unsatisfactory features of the Schwarzschild solution, notably the lack of agreement with Mach's Principle. It was shown in I that the two solutions found for Kilmister's equations were far less satisfactory than the Schwarzschild solution. However Kilmister's choice of equations was somewhat arbitrary and in this article we propose to derive field equations for the ennuple vectors from variational principles and to use the methods of I to find solutions. These field equations will be generalized field equations, rather than alternatives, since the Einstein field equations occur as a particular case.

Field equations of this type were the subject of an investigation by Pellegrini and Plebanski (1963, to be referred to as PP) but their main concern was to obtain Einstein's field equations, and hence the Schwarzschild solution, together with additional conditions which would enable them to find a satisfactory energy–momentum complex. Here we are concerned with the problem of finding solutions to the generalized field equations other than the Schwarzschild solution in the hope that one, at least, will have the good points of the Schwarzschild solution, that is, agreement with observation, with none of its difficulties an mentioned in I.

In fact, all the solutions that we find to these generalized field equations are inferior to the Schwarzschild solution, although they have a number of interesting features. This is disappointing but we feel that this investigation is necessary if one takes the point of view, expressed in PP and by Plebanski (1962), that the ennuple, rather than the metric tensor, should be regarded as the basic physical entity. If this view is accepted then there is no reason to assume that the Einstein field equations are necessarily superior to any other field equations to be considered here. The final choice of field equations may have to be based on the suitability of the solutions found to the various field equations. If this is so then the present investigation is inconclusive since only a restricted number of solutions have been found so far.

In § 2 we establish two theorems concerning the divergence of tensors obtained from variational principles in which the ennuple components $h_i{}^\alpha$ are the quantities to be varied and in § 3 the variational principle technique is used to find the generalized field equations. A discussion of the weak field approximations is given in § 4. In § 5 the field equations are solved in certain particular cases using the method of solution devised in I, and in § 6 the solutions are discussed.

Throughout this article greek letters are tensor suffixes and latin letters are ennuple, or tetrad, suffixes. Both types of suffix take the values 1, 2, 3, 4.

The notation used in I is as follows: The ennuple h_i^{α} and its inverse h^i_{α} satisfy

$$h_i^{\alpha} h_{j\alpha} = \eta_{ij}$$
$$h_{i\alpha} h^i_{\beta} = g_{\alpha\beta}$$

where η_{ij} is the Minkowski metric tensor $(-1, -1, -1, 1)$ which is used to raise and lower ennuple suffixes.

Two asymmetric affine connections can be defined by using the ennuple, namely

$$\Delta^{\alpha}_{\beta\gamma} = h_i^{\alpha} h^i_{\gamma,\beta} = -h^i_{\gamma} h_i^{\alpha}{}_{,\beta}$$

and

$$\Gamma^{\alpha}_{\beta\gamma} = h_i^{\alpha} h^i_{\beta,\gamma} = -h^i_{\beta} h_i^{\alpha}{}_{,\gamma}$$

so that

$$\Delta^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}.$$

Covariant differentiation with respect to $\Delta^{\alpha}_{\beta\gamma}$ is denoted by a single line $|$.

A tensor $C^{\alpha}_{\beta\gamma}$ is defined by

$$C^{\alpha}_{\beta\gamma} = \Delta^{\alpha}_{\beta\gamma} - \left\{ {\alpha \atop \beta\gamma} \right\} = h_i^{\alpha} h^i_{\gamma;\beta}$$

where the semicolon denotes covariant differentiation with respect to the Christoffel bracket connection $\left\{ {\alpha \atop \beta\gamma} \right\}$. If follows that when $C^{\alpha}_{\beta\gamma} = 0$, the space is flat; the converse is true only if we use the 'proper ennuple'. Note that $C_{\alpha\beta\gamma} = g_{\alpha\epsilon} C^{\epsilon}_{\beta\gamma}$ is antisymmetric in the first and third suffixes: From $C^{\alpha}_{\beta\gamma}$ we can form a vector $C_{\gamma} = C^{\alpha}_{\alpha\gamma} = -C_{\gamma\alpha}{}^{\alpha}$.

2. Divergence theorems

In the usual formulation of general relativity theory the field equations are obtained by an action principle in which the variation of a scalar density with respect to a small variation of the metric tensor $g_{\alpha\beta}$ is considered. It is well known that such a variation produces a symmetric tensor $B_{\alpha\beta}$ which is such that $B_{\alpha}{}^{\beta}{}_{;\beta} = 0$ (Eddington 1924, § 61). We will now establish the corresponding result for variations with respect to small variations of the ennuple h_i^{α}.

Let L be an invariant function of the $h_{i\alpha}$ and their derivatives and let h be the determinant of $h_{i\alpha}$. It follows that $h = \sqrt{-g}$. Define the invariant

$$J = \int_D Lh \, d^4x$$

where $d^4x = dx^1 \, dx^2 \, dx^3 \, dx^4$, over the domain D of space–time bounded by a hypersurface S. Consider the variation of J due to a small variation in the $h_{i\alpha}$ in such a way that the variation of the ennuple and its derivatives vanish on the boundary S. Then it is easily shown that

$$\delta J = \delta \int_D Lh \, d^4x = \int_D hP^{i\alpha} \delta h_{i\alpha} \, d^4x$$

where $P^{i\alpha}$ is the variational derivative of L with respect to $h_{i\alpha}$, that is

$$P^{i\alpha} = \frac{\delta L}{\delta h_{i\alpha}}.$$

If we suppose that the variation $\delta h_{i\nu}$ arises merely from a coordinate transformation then an argument analogous to that given by Eddington (1924 § 61) leads to

$$\delta J = \int_D \left((hP^{i\alpha}h_{i\beta})_{,\alpha} - hP^{i\alpha}\frac{\partial h_{i\alpha}}{\partial x^\beta} \right) \delta x^\beta \, \mathrm{d}^4x.$$

This vanishes under arbitrary variations δx^β so

$$(hP^{i\alpha}h_{i\beta})_{,\alpha} - hP^{i\alpha}\frac{\partial h_{i\alpha}}{\partial x^\beta} = 0.$$

If we write $P_\beta{}^\alpha = h_{i\beta}P^{i\alpha}$ this becomes

$$(hP_\beta{}^\alpha)_{,\alpha} - hP^{i\alpha}h_{i\gamma}\Delta^\gamma{}_{\beta\alpha} = 0$$

that is

$$h(P_\beta{}^\alpha{}_{,\alpha} + \{^{\gamma}_{\alpha\gamma}\}P_\beta{}^\alpha - P_\gamma{}^\alpha\Delta^\gamma{}_{\beta\alpha}) = 0 \tag{1}$$

that is

$$P_\beta{}^\alpha|_\alpha = 0 \tag{2}$$

since $\Delta^\gamma_{\alpha\gamma} - \{^{\gamma}_{\alpha\gamma}\} = C^\gamma{}_{\alpha\gamma} = 0$. Hence we have *theorem* 1:

The variational derivative

$$P^{i\alpha} = \frac{\delta L}{\delta h_{i\alpha}}$$

of an invariant function L of the $h_{i\alpha}$ and their derivatives satisfies the vanishing divergence equation

$$P_\beta{}^\alpha|_\alpha = 0$$

where $P_\beta{}^\alpha = h_{i\beta}P^{i\alpha}$ and the line denotes covariant differentiation with respect to the asymmetric affine connection $\Delta^\alpha{}_{\beta\gamma} = h_i{}^\alpha h^i{}_{\gamma,\beta}$.

Note that it does not follow that $P^{\beta\alpha}{}_{|\alpha} = 0$ since $g_{\alpha\beta|\gamma} \neq 0$.

If the Lagrangian density of matter, including all nongravitational fields, is added to L then, in the usual way, the field equations resulting from the action principle will be

$$P^{i\alpha} = kT^{i\alpha}$$

that is

$$P_\beta{}^\alpha = kT_\beta{}^\alpha \tag{3}$$

where k is a constant and $T_\beta{}^\alpha$ is the energy–momentum tensor of matter and other fields. It follows from (2) that $T_\beta{}^\alpha$ satisfies

$$T_\beta{}^\alpha|_\alpha = 0. \tag{4}$$

In general relativity the usual conservation law $T_\beta{}^\alpha{}_{;\alpha} = 0$ is imposed as a covariant generalization of the law $T_\beta{}^\alpha{}_{,\alpha} = 0$ in special relativity. Equation (4) is a different covariant generalization of the special relativistic law; in the absence of gravitation, that is in a flat space–time, it is easily seen from (1) that (4) reduces to $T_{\beta}{}^{\alpha}{}_{,\alpha} = 0$ if we use cartesian coordinates and the proper ennuple.

Now equation (1) can be written in the form

$$P_{\beta}{}^{\alpha}{}_{,\alpha} + \left\{ {\gamma \atop \alpha\gamma} \right\} P_{\beta}{}^{\alpha} - \left\{ {\gamma \atop \beta\alpha} \right\} P_{\gamma}{}^{\alpha} - P^{\gamma\alpha} C_{\gamma\beta\alpha} = 0$$

and since $C_{\gamma\beta\alpha}$ is antisymmetric in γ, α it follows that the last term is zero if $P^{\beta\alpha}$ is a symmetric tensor. Hence we have *theorem* 2:

If $P^{\beta\alpha}$ is a symmetric tensor then it satisfies the divergence equation

$$P_{\beta}{}^{\alpha}{}_{;\alpha} = 0.$$

Hence if, for the usual reasons, the energy–momentum tensor is required to be symmetric, it will satisfy the covariant divergence equation $T_{\beta}{}^{\alpha}{}_{;\alpha} = 0$. The field equations (3) will then have the form

$$\left. \begin{aligned} P^{[\beta\alpha]} &= 0 \\ P^{(\beta\alpha)} &= kT^{\beta\alpha} \end{aligned} \right\} \tag{5}$$

where the square and curved brackets denote the antisymmetric and symmetric parts respectively.

3. Derivation of the field equations

In PP, invariants bilinear in the first derivatives of the ennuple field are constructed and a linear combination of them is used as the most general Lagrangian L. It is found that there are only seven possible invariants one of which is a pure divergence and, of the others, only four are independent, namely

$$\begin{aligned} L_1 &= C^{\alpha\beta\gamma} C_{\alpha\beta\gamma} \\ L_2 &= C^{\alpha\beta\gamma} C_{\alpha\gamma\beta} \\ L_3 &= C^{\alpha} C_{\alpha} \\ L_4 &= \epsilon^{\alpha\beta\gamma\delta} C_{\alpha} C_{\beta\gamma\delta} \end{aligned}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi–Civita tensor density. (These are not the four invariants used in PP but are linear combinations of them which give simple expressions in our notation.)

By carrying out the variations

$$\delta \int L_a h \, \mathrm{d}^4 x = 0 \qquad (a = 1, \ldots, 4)$$

where the ennuple components $h^{i}{}_{\alpha}$ are the quantities to be varied, we find the following field equations corresponding to L_1, L_2, L_3, L_4 respectively

$$D_{\mu}{}^{\nu} = k_1 T_{\mu}{}^{\nu} \tag{6}$$

$$E_{\mu}{}^{\nu} = k_2 T_{\mu}{}^{\nu} \tag{7}$$

$$F_{\mu}{}^{\nu} = k_3 T_{\mu}{}^{\nu} \tag{8}$$

$$H_{\mu}{}^{\nu} = k_4 T_{\mu}{}^{\nu}. \tag{9}$$

The expressions on the left sides of these equations are

$$D_\mu{}^\nu = \delta^\nu_\mu C^{\alpha\beta\gamma}C_{\alpha\beta\gamma} - 2C_{\alpha\mu\gamma}C^{\alpha\nu\gamma} - 2(C_\mu{}^{\alpha\nu} - C^\nu{}_\mu{}^\alpha + C^{\alpha\nu}{}_\mu)_{;\alpha} \tag{10}$$

$$E_\mu{}^\nu = \delta^\nu_\mu C^{\alpha\beta\gamma}C_{\alpha\gamma\beta} - 2C^\alpha{}_\mu{}^\nu{}_{;\alpha} - 2C_{\alpha\mu\beta}C^{\alpha\beta\nu} \tag{11}$$

$$F_\mu{}^\nu = \delta^\nu_\mu(C^\alpha C_\alpha + 2C^\alpha{}_{;\alpha}) + 2C_\alpha C^\alpha{}_\mu{}^\nu - 2C^\nu{}_{;\mu} \tag{12}$$

$$\begin{aligned} H_\mu{}^\nu = {} & (\eta^{\nu\beta\gamma\delta}C_{\beta\gamma\delta})_{;\mu} + \delta^\nu_\mu\eta^{\alpha\beta\gamma\delta}C_\alpha C_{\beta\gamma\delta} - g_{\mu\beta}\eta^{\nu\beta\gamma\delta}C_{\gamma;\delta} \\ & - 2\eta^{\nu\beta\gamma\delta}C_\mu C_{\beta\gamma\delta} + 2\eta^{\nu\beta\gamma\delta}C_\beta C_{\mu\gamma\delta} + \eta^{\alpha\beta\gamma\delta}C_{\beta\gamma\delta}C^\nu{}_{\mu\alpha} \end{aligned} \tag{13}$$

where the tensor $\eta^{\alpha\beta\gamma\delta}$ is defined by

$$\eta^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta} = \frac{1}{h}\epsilon^{\alpha\beta\gamma\delta}.$$

The most general field equations will thus be of the form

$$a_1 D_\mu{}^\nu + a_2 E_\mu{}^\nu + a_3 F_\mu{}^\nu + a_4 H_\mu{}^\nu = kT_\mu{}^\nu \tag{14}$$

where a_1, a_2, a_3, a_4 are constants.

Note that the Ricci scalar is given in terms of $C^\alpha{}_{\beta\gamma}$ by

$$R = C^{\alpha\beta\gamma}C_{\alpha\gamma\beta} - C^\alpha C_\alpha - 2C^\alpha{}_{;\alpha}. \tag{15}$$

If we consider the variation

$$\delta \int Rh \, d^4x$$

then the last term of (15) contributes nothing since it is a divergence and we obtain

$$-2G^\nu_\mu = E^\nu_\mu - F^\nu_\mu \tag{16}$$

where G^ν_μ is the Einstein tensor. Thus, as expected, the usual field equations of general relativity are a special case of the field equations (14).

4. Weak field approximations

Before attempting to find solutions of the field equations we will investigate their weak-field approximations. For this purpose we follow Kilmister in using an almost cartesian system in an almost Minskowskian space–time and choosing the ennuple to be (almost) the simplest diagonal form, that is, we choose it to be (almost) the proper ennuple. In this case we may write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa_{\alpha\beta} \tag{17}$$

and

$$h_{i\alpha} = \eta_{i\alpha} + \epsilon_{i\alpha} \tag{18}$$

where both $\kappa_{\alpha\beta}$ and $\epsilon_{i\alpha}$ are small and $\kappa_{\alpha\beta}$ is symmetric. This choice of ennuple ensures that $C^\alpha{}_{\beta\gamma}$, which is zero for the proper ennuple in a flat space, is small in the almost flat space.

We then have

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \kappa^{\alpha\beta}$$

from which, to the first order in small quantities

$$\kappa^{\alpha\beta} = \eta^{\alpha\delta}\eta^{\beta\epsilon}\kappa_{\delta\epsilon}.$$

As usual we impose the de Donder condition

$$g^{\alpha\beta}\left\{{\gamma \atop \alpha\beta}\right\} = 0$$

which leads to

$$\kappa^{\beta\gamma}{}_{,\beta} = \tfrac{1}{2}\eta^{\gamma\epsilon}\kappa^{\alpha}{}_{\alpha,\epsilon}. \tag{19}$$

The ennuple must satisfy

$$\eta^{ij}h_{i\alpha}h_{j\beta} = \eta_{\alpha\beta} + \kappa_{\alpha\beta}$$

so from (18) we find that

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = \kappa_{\alpha\beta} \tag{20}$$

where $\epsilon_{\alpha\beta} = \delta^{i}{}_{\alpha}\epsilon_{i\beta}$. We assume, as in I, that the ennuple corresponding to the metric tensor of a static space–time will also be time-independent.

Applying this to the field tensor $D_{\mu}{}^{\nu}$ given by (10) we find that

$$D_{\mu\nu} = -2\Box\kappa_{\mu\nu} - \Box\epsilon_{\mu\nu} + \Box\epsilon_{\nu\mu} - 3\epsilon_{\mu}{}^{\alpha}{}_{,\alpha\nu} - \epsilon_{\nu}{}^{\alpha}{}_{,\alpha\mu} + \epsilon^{\alpha}{}_{\nu,\alpha\mu} - \epsilon^{\alpha}{}_{\mu,\alpha\nu}.$$

For a static metric this gives

$$D_{44} = 2\nabla^{2}\kappa_{44}$$

so that the vacuum field equations $D_{\mu\nu} = 0$ imply that Laplace's equation is satisfied if we make the usual identification $\kappa_{44} = 2\phi/c^{2}$.

Consider now the equations $D_{\mu\nu} = k_{1}T_{\mu\nu}$. In a static space–time, with matter but no internal forces we have

$$T_{4}^{4} = \rho c^{2} \text{ and } T_{\alpha}^{\beta} = 0 \qquad (\alpha, \beta \text{ not both } 4).$$

For a weak field and almost cartesian coordinates the only covariant component, to the first order, is $T_{44} = \rho c^{2}$. Then $D_{44} = k_{1}T_{44}$ implies

$$\frac{4}{c^{2}}\nabla^{2}\phi = k_{1}\rho c^{2}$$

and this is Poisson's equation $\nabla^{2}\phi = 4\pi\gamma\rho$ if $k_{1} = 16\pi\gamma/c^{4}$ so that field equations are

$$D_{\mu\nu} = \frac{16\pi\gamma}{c^{4}}T_{\mu\nu}.$$

By an identical argument we find that equation (7) gives Laplace's equation in the absence of matter and Poisson's equation in matter provided that we make the identification

$$E_{\mu\nu} = \frac{8\pi\gamma}{c^{4}}T_{\mu\nu}.$$

Equation (8) leads to $\Box\kappa = 0$ in vacuo which imposes no restriction on κ_{44} so this equation does not lead to either Laplace's equation or Poisson's equation. Equation (9) is identically zero in the weak-field approximation.

Hence we have that the field equations (6), (7) are the only ones which individually can be used to describe the gravitational field. It follows that in the general field equations (14) we must have either $a_1 \neq 0$ or $a_2 \neq 0$ for the equations to be suitable for the description of the gravitational field.

In the next section we will be concerned with the field equation

$$E_{\mu\nu} - aF_{\mu\nu} = kT_{\mu\nu} \tag{21}$$

where a is a constant. We will now investigate the weak field approximation of these equations. From (16) equations (21) can be written in the form

$$G_{\mu\nu} - \tfrac{1}{2}(1-a)F_{\mu\nu} = -\tfrac{1}{2}kT_{\mu\nu} \tag{22}$$

so that when $a = 1$ we have the Einstein equations. For a weak field (21) becomes

$$E_{\mu\nu} - aF_{\mu\nu} = -\Box\kappa_{\mu\nu} + \epsilon_{\mu}{}^{\alpha}{}_{,\alpha\nu} + \epsilon_{\nu}{}^{\alpha}{}_{,\alpha\mu} + \epsilon^{\alpha}{}_{\mu,\alpha\nu} - \epsilon^{\alpha}{}_{\nu,\alpha\mu}$$
$$- a\eta_{\mu\nu}(2\epsilon^{\alpha\beta}{}_{,\alpha\beta} - \Box\kappa) + a(2\epsilon^{\alpha}{}_{\nu,\alpha\mu} - \kappa_{,\nu\mu}).$$

Contracting this expression and using (20) we obtain for a vacuum field

$$(1-3a)\Box\kappa - (1-3a)\kappa^{\alpha\beta}{}_{,\alpha\beta} = 0 \tag{23}$$

so that, if $a \neq \frac{1}{3}$, equation (19) gives

$$\Box\kappa = \kappa^{\alpha\beta}{}_{,\alpha\beta} = 0.$$

Taking the (4, 4) component of (21) for a static vacuum field

$$E_{44} - aF_{44} \equiv -\Box\kappa_{44} - 2a\epsilon^{\alpha\beta}{}_{,\alpha\beta} + a\Box\kappa = 0$$

that is

$$-\Box\kappa_{44} + \tfrac{1}{2}a\Box\kappa = 0 \tag{24}$$

that is

$$\Box\kappa_{44} = 0 \tag{25}$$

so Laplace's equation is satisfied.

In matter $T_{44} = T_{\mu}{}^{\mu} = \rho c^2$ to the first order and we find

$$(1-3a)(\Box\kappa - \kappa^{\alpha\beta}{}_{,\alpha\beta}) = k\rho c^2 \tag{26}$$

from which we obtain, if $a \neq \frac{1}{3}$

$$-\Box\kappa_{44} = \frac{2a-1}{3a-1}k\rho c^2 \tag{27}$$

that is

$$\nabla^2\phi = \tfrac{1}{2}k\left(\frac{2a-1}{3a-1}\right)\rho c^4.$$

This is Poisson's equation provided that

$$k = \frac{8\pi\gamma}{c^4}\frac{3a-1}{2a-1}. \tag{28}$$

There are two special cases to consider, namely $a = \frac{1}{2}$ and $a = \frac{1}{3}$. When $a = \frac{1}{2}$ we see from (27) that we can obtain Laplace's equation but not Poisson's equation, so that the field equations with $a = \frac{1}{2}$ can describe a vacuum field but not a material distribution. When $a = \frac{1}{3}$ we see from (26) that $k = 0$, so again the field equations cannot describe a matter distribution. For a vacuum field we cannot find the relation (25) since (23) vanishes identically and we are left with only the equation (24), that is

$$-\Box\kappa_{44}+\tfrac{1}{6}\Box\kappa = 0.$$

It follows that the field equations with $a = \frac{1}{3}$ reduce to Laplace's equation if, and only if, the additional restriction $\Box\kappa = 0$ is applied.

Since in this article we shall be concerned only with vacuum solutions of the field equations, the two cases $a = \frac{1}{2}$ and $a = \frac{1}{3}$ will be considered as being acceptable.

5. Explicit form of the field equations

Consider first the general field equations (14) in a static spherically symmetric space–time of the form

$$ds^2 = e^{2\mu}\, dt^2 - e^{2\nu}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{29}$$

where μ, ν are functions of r only and the velocity of light has been given unit value. Following the argument of I we use an ennuple of the form

$$h_i{}^\alpha = \Theta\Phi\mathscr{F}$$

where Θ is the matrix

$$\begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Φ is the product

$$H_{34}(\beta_3)H_{24}(\beta_2)H_{14}(\beta_1)$$

in that order of the matrix

$$H_{14}(\beta_1) = \begin{bmatrix} \cosh\beta_1 & 0 & 0 & \sinh\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_1 & 0 & 0 & \cosh\beta_1 \end{bmatrix}$$

representing rotations in the (x, t) plane and two similar matrices $H_{24}(\beta_2)$, $H_{34}(\beta_3)$ representing rotations in the (y, t) and (z, t) planes and $\mathscr{F}$ is the matrix

$$\mathscr{F} = \begin{bmatrix} e^{-\nu} & & & \\ & \frac{1}{r}e^{-\nu} & & 0 \\ & & \frac{1}{r\sin\theta}e^{-\nu} & \\ 0 & & & e^{-\mu} \end{bmatrix}.$$

For a static space–time we assume that β_1, β_2, β_3 are functions of r only. The reasons for adopting an ennuple of this form are explained at length in I.

With this tetrad the nonzero components of the tensor $C^{\alpha}{}_{\beta\gamma}$ are found to be

$$C^1{}_{12},\, C^1{}_{22},\, C^1{}_{13},\, C^1{}_{23},\, C^1{}_{33},\, C^1{}_{14},\, C^1{}_{24},\, C^1{}_{34},\, C^1{}_{44},$$
$$C^2{}_{11},\, C^2{}_{21},\, C^2{}_{13},\, C^2{}_{23},\, C^2{}_{33},\, C^2{}_{14},\, C^2{}_{24},\, C^2{}_{34},$$
$$C^3{}_{11},\, C^3{}_{21},\, C^3{}_{31},\, C^3{}_{12},\, C^3{}_{22},\, C^3{}_{32},\, C^3{}_{14},\, C^3{}_{34},$$
$$C^4{}_{11},\, C^4{}_{21},\, C^4{}_{31},\, C^4{}_{41},\, C^4{}_{12},\, C^4{}_{22},\, C^4{}_{32},\, C^4{}_{13},\, C^4{}_{33}.$$

We can now compute the sixteen vacuum field equations (14) from the values of $C^{\alpha}{}_{\beta\gamma}$. These field equations are complicated; however after a tedious but straightforward calculation using the (2, 2), (2, 3) and (2, 4) equations we find that

$$\beta_2 = \beta_3 = 0$$

as in I. Putting β_2, β_3 equal to zero the following results are found:

$$H_{\mu\nu} \equiv 0 \tag{30}$$

and

$$C^{\alpha\beta\gamma}C_{\alpha\beta\gamma} = 2C^{\alpha\beta\gamma}C_{\alpha\gamma\beta}. \tag{31}$$

The second result means that

$$D_{\mu\nu} = 2E_{\mu\nu} \tag{32}$$

and it follows from (30) and (32) that the general field equations (14) simplify to the form (21), that is

$$E_{\mu\nu} - aF_{\mu\nu} = kT_{\mu\nu}$$

where k is given by (28), or to the equivalent equation (22).

Note that the relation (31) is not an identity and appears to be due entirely to the choice of ennuple. This choice of ennuple thus leads to a remarkable simplification of the field equations.

In a vacuum field the equations (22) become

$$G_{\mu\nu} - \tfrac{1}{2}(1-a)F_{\mu\nu} = 0$$

which can be written as

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - (1-a)\{C_{\alpha}C^{\alpha}{}_{\mu\nu} - C_{\nu;\mu} + \tfrac{1}{2}g_{\mu\nu}(C^{\alpha}C_{\alpha} + 2C^{\alpha}{}_{;\alpha})\} = 0 \tag{33}$$

from which we obtain the field equations in the form

$$R_{\mu\nu} = (1-a)(C_{\alpha}C^{\alpha}{}_{\mu\nu} - C_{\nu;\mu} - \tfrac{1}{2}g_{\mu\nu}C^{\alpha}{}_{;\alpha}). \tag{34}$$

The only independent field equations which are not identically zero are the (1, 1), (2, 2), (4, 4) and (1, 4) equations which are respectively:

$$\begin{aligned}&\tfrac{1}{2}(3a-1)\mu'' + (3a-1)\nu'' + \tfrac{1}{2}(a+1)\mu'^2 + \tfrac{1}{2}(a-3)\mu'\nu' + (1-a)\nu'^2 + 2(a-1)\frac{\mu'}{r}\\ &\quad + (a+1)\frac{\nu'}{r} + (1-a)\frac{1}{r^2} + (1-a)\frac{\mu'}{r}\cosh\beta_1 + (a-1)\frac{\nu'}{r}\cosh\beta_1 + (a-1)\beta_1'^2\\ &\quad + (1-a)\frac{1}{r}\beta_1'\sinh\beta_1 + (a-1)\frac{1}{r^2}\cosh\beta_1 = 0\end{aligned} \tag{35}$$

$$\tfrac{1}{2}(a-1)\mu''+a\nu''+\tfrac{1}{2}(a-1)\mu'^2+\tfrac{1}{2}(3a-1)\mu'\nu'+a\nu'^2+(2a-1)\frac{\mu'}{r}+3a\frac{\nu'}{r}+(1-a)\frac{1}{r^2}$$
$$+(a-1)\frac{1}{r^2}\cosh\beta_1+(a-1)\frac{\nu'}{r}\cosh\beta_1 = 0 \tag{36}$$

$$\tfrac{1}{2}(a+1)\mu''+(a-1)\nu''+\tfrac{1}{2}(a+1)\mu'^2+\tfrac{1}{2}(3a-1)\mu'\nu'+(a-1)\nu'^2+2a\frac{\mu'}{r}+3(a-1)\frac{\nu'}{r}$$
$$+(a-1)\frac{1}{r^2}+(1-a)\frac{\mu'}{r}\cosh\beta_1+(1-a)\frac{\nu'}{r}\cosh\beta_1+(1-a)\frac{1}{r^2}\cosh\beta_1$$
$$+(1-a)\frac{1}{r}\beta_1'\sinh\beta_1 = 0 \tag{37}$$

$$(a-1)\left\{\beta_1''+\left(\mu'+\nu'+\frac{2}{r}\right)\beta_1'-\frac{2}{r}\left(\nu'+\frac{1}{r}\right)\sinh\beta_1\right\} = 0. \tag{38}$$

No general solution of these equations has been found, so, as an additional simplification, we will seek only those solutions for which $\beta_1 = 0$, that is, in the terminology of I, we will seek solutions that are *reducible* to zero-parameter solutions. Putting $\beta_1 = 0$ equations (35) to (37) become:

$$\tfrac{1}{2}(3a-1)\mu''+(3a-1)\nu''+\tfrac{1}{2}(a+1)\mu'^2+\tfrac{1}{2}(a-3)\mu'\nu'+(1-a)\nu'^2+(a-1)\frac{\mu'}{r}+2a\frac{\nu'}{r} = 0 \tag{39}$$

$$\tfrac{1}{2}(a-1)\mu''+a\nu''+\tfrac{1}{2}(a-1)\mu'^2+\tfrac{1}{2}(3a-1)\mu'\nu'+a\nu'^2+(2a-1)\frac{\mu'}{r}+(4a-1)\frac{\nu'}{r} = 0 \tag{40}$$

$$\tfrac{1}{2}(a+1)\mu''+(a-1)\nu''+\tfrac{1}{2}(a+1)\mu'^2+\tfrac{1}{2}(3a-1)\mu'\nu'+(a-1)\nu'^2+(a+1)\frac{\mu'}{r}+2(a-1)\frac{\nu}{r} = 0. \tag{41}$$

Note that when $a = 1$ the equations (34) become the Einstein vacuum field equations which give the Schwarzschild solution. In this case (38) vanishes identically for all values of β_1 which illustrates the point that in this case the Lagrangian and the resulting field equations are functions of the metric tensor rather than functions of the ennuple. We seek solutions of equations (39) to (41) for which $a \neq 1$.

6. Solutions of the field equations

The following solutions of equations (39) to (41) have been found:

(i) $a = 0$:

(A)
$$e^{2\mu} = \left(1+\frac{m}{r}+\frac{3m^2}{8r^2}\right)^{4/3}\exp\left\{\frac{4\sqrt{2}}{3}\tan^{-1}\left(\frac{\sqrt{2}m/4r}{1+(m/2r)}\right)\right\}$$
$$e^{2\nu} = \left(1+\frac{m}{r}+\frac{3m^2}{8r^2}\right)^{2/3}\exp\left\{\frac{-4\sqrt{2}}{3}\tan^{-1}\left(\frac{\sqrt{2}m/4r}{1+(m/2r)}\right)\right\}$$

where m is a constant.

(B) $e^{2\mu} = r^{-8/3}$ $\qquad e^{2\nu} = r^{-4/3}$.

(ii) $a = \frac{1}{3}$:

$$\mu = \nu.$$

(iii) $a = \frac{11}{25}$:

(A) $$e^{2\mu} = \left(1 - \frac{m}{2r}\right)^{4} \qquad e^{2\nu} = \left(1 - \frac{m}{2r}\right)^{-2}$$

(B) $$e^{2\mu} = r^{-7/4} \qquad e^{2\nu} = r^{-9/4}.$$

(iv) $a = \frac{1}{2}$:

(A) $$e^{2\mu} = \left(1 + \frac{b}{r}\right)^{8/5} \qquad e^{2\nu} = \frac{1}{r^2}\left(1 + \frac{b}{r}\right)^{2/5}$$

where b is a constant.

(B) $$\mu = 0 \qquad e^{2\nu} = \left(1 + \frac{m}{r}\right)^{2}$$

where m is a constant.

(C) $$\mu = 0 \qquad e^{2\nu} = \frac{b^2}{r^2}$$

where b is a constant.

(D) $$e^{2\mu} = r^{-8/5} \qquad e^{2\nu} = r^{-12/5}.$$

(v) $a = 3$:

$$e^{2\mu} = e^{-m/r}\left(1 - \frac{2m}{5r}\right)^{5/2} \qquad e^{2\nu} = e^{m/r}\left(1 - \frac{2m}{5r}\right)^{-1/2}.$$

These solutions may be grouped as follows:

Type I. *Conformally flat space–times*

From (ii), when $a = \frac{1}{3}$ the solutions are conformally flat space–times. If we consider the field equations for the case when μ, ν are each functions of r and t, then we find that the additional terms in equations (39) to (41) each have a factor $(1-3a)$, so that when $a = \frac{1}{3}$ the time-dependent solutions are again conformally flat space–times. Since any spherically symmetric conformally flat space–time can be written in the form

$$ds^2 = e^{2\mu}(dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2)$$

where $\mu = \mu(r, t)$, it follows that every spherically symmetric conformally flat space–time is a solution of the field equations (22). This result must be due to the choice of ennuple since the equations (22) with $a = \frac{1}{3}$ are not sufficient to ensure the vanishing of the Weyl tensor.

It is easily shown that equations (39) to (41) admit conformally flat solutions only when $a = \frac{1}{3}$, and it can also be shown that if we use equations (35) to (38) taking $\beta_1 \neq 0$, then the same results appear when we use the fact a space–time of the form (29) with $\mu = \mu(r, t)$, $\nu = \nu(r, t)$ is conformally flat if

$$\exp(\mu - \nu) = a + br^2$$

where a, b are functions of t only. Hence we have the curious result that when $a = \frac{1}{3}$ an ennuple can always be found such that every conformally flat spherically symmetric space–time is a solution of the generalized field equations and this ennuple gives conformally flat solutions only when $a = \frac{1}{3}$.

Amongst these conformally flat solutions are the static Robertson–Walker models

$$ds^2 = dt^2 - \frac{R^2}{(1+kr^2/4)^2}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2).$$

When $k = +1$, this is the Einstein universe and when $k = -1$ this is an open model. The latter was found in I as a solution of Kilmister's equations $C^{\alpha}{}_{\beta\gamma;\alpha} = 0$ and, as in I, these are solutions of the field equations if $\beta_1 \neq 0$ or if the space–time is taken to be nonstatic. Since the Robertson–Walker metric corresponds to a homogeneous, isotropic space–time it follows that solutions both represent curved empty space–times, in disagreement with Mach's principle.

Two other conformally flat solutions are worth noting. Firstly the space–time

$$ds^2 = \left(1 - \frac{m}{r}\right)^2 (dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2) \tag{42}$$

which arises in scalar gravitational theories (see Schild 1962) and secondly

$$ds^2 = \frac{a^2}{r^2}(dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2)$$

which is the solution of the Einstein–Maxwell equations discovered by Robinson (1959) and which has been shown to represent the field of a repelling particle by Lovelock (1967). Hence the generalized field equations admit solutions with nongravitational characteristics.

Type II. *Asymptotically flat space–times*

Solutions (iA), (iiiA), (ivB), (v) are of this type as well as some solutions of type I, such as (42). It can be shown (Phillips 1969) that unless $a = 1$, which gives the Schwarzschild solution, no solution of equations (39) to (41) can agree with the Schwarzschild solution in regard to the three tests of general relativity. If we denote the values given by the Schwarzschild solution for the three tests by unity then the proportionate values given by the solutions found here are

Solution	Advance of perihelion	Light-ray deflection	Red shift
(iA)	$\frac{11}{24}$	$\frac{1}{2}$	1
(ii, 42)	$-\frac{1}{6}$	0	1
(iiiA)	$\frac{3}{4}$	$\frac{3}{4}$	1
(ivB)	0	$\frac{1}{2}$	0
(v)	$\frac{11}{15}$	$\frac{4}{5}$	1

Type III. *Recurrent space–times*

Solutions (iB), (iiiB) and (ivD) are found to be recurrent space–times, that is, they satisfy the relation

$$R_{\alpha\beta\gamma\delta;\epsilon} = \kappa_{\epsilon} R_{\alpha\beta\gamma\delta}$$

where the recurrence vector κ_ϵ is given by

$$\kappa_\epsilon = \left[-\frac{2m}{r}, 0, 0, 0\right] \tag{43}$$

where

$$m = \frac{3a-1}{a-3}. \tag{44}$$

In fact, it is found that for any value of a other than $a = 1$ and $a = 3$, there is a recurrent space–time solution of equations (39) to (41) with recurrence vector given by (43). These space–times are of the form

$$ds^2 = r^{-2(m+1)}\,dt^2 - r^{2(m-1)}(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2)$$

where m is given by (44).

The physical interpretation, if any, of these solutions is quite obscure; they cannot represent the gravitational field of a massive body since the orbits of particles are not approximately ellipses.

Type IV

Solution (ivc) is precisely the zero-parameter solution of Kilmister's equations found and discussed in I. It was shown that some of the properties of this space–time were consistent with an empty space–time with curvature due to the presence of distant matter, but some of its other properties did not admit a reasonable physical interpretation.

Type V

Solution (ivA) does not fall into any of the above categories. The orbits of particles are not approximately ellipses and the physical interpretation is obscure.

7. Conclusions

One of the significant features of the generalized field equations formulated here is the embarrassment of riches provided by the large number of solutions that have been found so far. It is particularly noticeable that the largest numbers of solutions have been found for the two values, $a = \frac{1}{3}$ and $a = \frac{1}{2}$, for which the field equations can only describe vacuum fields.

Many of the solutions found here can be rejected on the grounds that they cannot describe the gravitational field of a massive body. Those that can describe such a field predict results for the three tests which differ considerably from those of the Schwarzschild solution. However, the solutions found here are presumably only a few of the many possible solutions that can be obtained for different values of a and different ennuples, so it is possible that one solution may exist that agrees with observational evidence without having those properties of the Schwarzschild solution to which Kilmister objected. These generalized field equations are aesthetically less satisfying than the Einstein equations since there is no unique solution and the choice of solution will have to be made on the observational evidence, but they are none the worse for that.

One question that we have been unable to answer and which may be worth considering is: Is there any value of a, other than $a = 1$, which leads to a unique solution for the field equations—even for the equations (39) to (41) obtained from the restricted ennuple?

Acknowledgments

The major portion of this work was done when BOJT was a member of the Department of Mathematics, University of Exeter; the remainder of his contribution was supported in part by the National Research Council of Canada through Operating Grant No. A7589. GWP is indebted to the Science Research Council for the award of a Research Studentship.

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